A water-based proof of the Cauchy–Schwartz inequality

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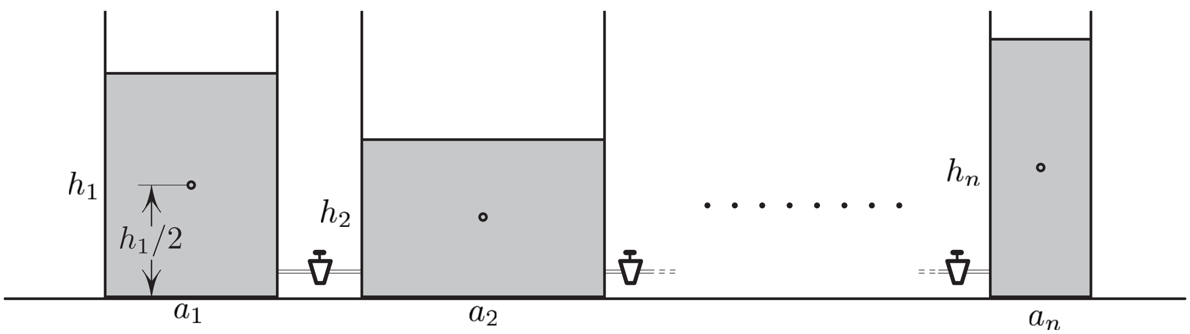
The standard proof of the inequality of the title may appear a bit dry, and so I am offering one that uses water. Consider n cylindrical cans with different crossectional areas a_k; fill the kth can to an arbitrarily chosen height h_k. Then open the valves, letting the water level out; the potential energy becomes less: $P_{\text{new}} \le P_{\text{old}}$, with the equality holding if and only if all the heights are equal at the outset. I claim that this amounts to the Cauchy–Schwartz inequality. What follows is a verification of this claim.



The equalized level is given by $\overline{h} = \Sigma a_k h_k / \Sigma a_k$, and the new, smaller potential energy is $\frac{1}{2}(\Sigma a_k)\overline{h}^2 = \frac{1}{2}(\Sigma a_k h_k)^2/\Sigma a_k$. So $P_{\text{new}} \le P_{\text{old}}$ amounts to

$$(\Sigma a_k h_k)^2/\Sigma a_k \le \Sigma a_k h_k^2, \quad \text{or} \quad (\Sigma a_k h_k)^2 \le \Sigma a_k h_k^2 \; \Sigma a_k. \tag{1}$$

This is the Cauchy-Schwartz in disguise. Indeed, driven by the desire to have $\Sigma x_k^2 \; \Sigma y_k^2$ on the right in (1), define $x_k, \; y_k$ by

$$a_k h_k^2 = x_k^2, \; a_k = y_k^2; \tag{2}$$

but then multiplying these two definitions (2) side-by-side ("sidewise"?) gives $a_k^2 h_k^2 = x_k^2 y_k^2$, or

$$a_k h_k = x_k y_k,$$

so that (1) indeed amounts to the Cauchy–Schwartz inequality

$$(\Sigma x_k y_k)^2 \le \Sigma x_k^2 \; \Sigma y_k^2.$$

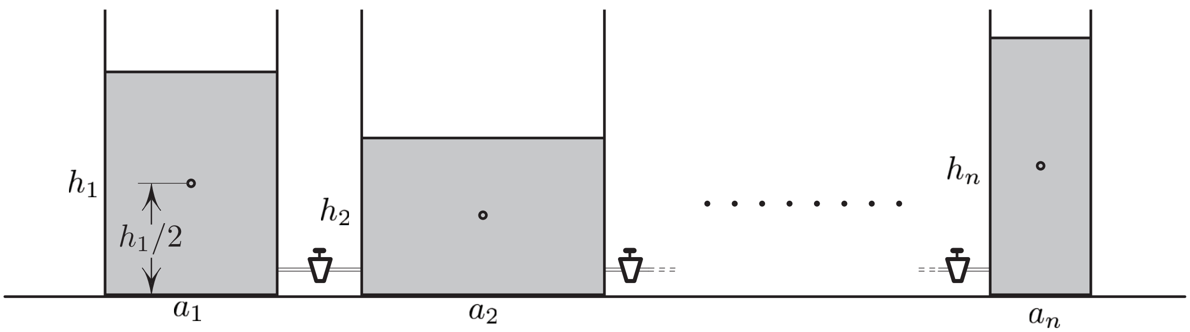


Figure 1: Opening the valves decreases the potential energy; this is equivalent to the Cauchy–Schwartz inequality.